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THEOREMS OF CONVERGENCE FOR MINIMAL SEQUENCES IN LIMIT ANALYSIS

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Abstract—Some fundamental existence and uniqueness theorems for the minimum of non-linear functionals are extended to limit analysis. The convergence of the Ritz Method is then examined, in particular: (a) conditions under which the approximations constitute a minimizing sequence; (b) stability of the numerical method; (c) estimate of the degree of convergence. An example of a finite–dimensional system illustrates the main results.

1. INTRODUCTION

ALTHOUGH many recent papers have been dedicated to the construction of kinematically admissible rate-fields in applying the upper bound theorem of limit analysis either for three dimensional bodies or for structures, the interest of justifying theoretically the legitimacy of such methods and of estimating their degree of convergence, seems not to be large at present.

In spite of that similar questions possess a now classical formulation for linear boundary value problems both with regard to the existence of the minima (see e.g. Weinberger [15] and Mikhlin [7]) and as to the convergence of the minimal sequences (see e.g. Mikhlin [7], Chapter III and Synge [14]). In more recent works by Mikhlin [9], [10] these results are extended to non-linear functionals connected with finite plasticity, in which those expressing the plastic power of the limit analysis can be mentioned.

This paper deals with the essential points of the variational problem for such functionals, i.e. the existence of the minimum, the convergence of the Ritz method, the estimation of the error of approximate solutions and the numerical solution of non-linear Ritz systems. All these results are applications of other more general considerations of Mikhlin [10]. A useful conclusion resulting from the treatment of non-linear functionals is that the existence and uniqueness of solution can be expected only for work-hardening solids. This fact was established on the basis of different considerations also by Koiter ([5], 5.4).

2. DEFINITIONS

We consider the limit analysis of a body V whose surface B is composed of two parts: B_1, on which velocities are zero; B_2, on which the surface tractions $\mathbf{T}$ are prescribed. The material is supposed to be rigid-work hardening, characterized by a convex resistance domain.

If $\mathbf{T}$ are the loads of incipient plasticization, we denote by $\mathbf{S}$ the actual stress and by $\dot{\mathbf{E}}$ the strain rate fields in the limit state, the latter deriving from a velocity field $\mathbf{v}$. At the beginning of the plastic flow, **S** reaches the yield surface of equation (see Jaunzemis [3], p. 21):

$$f(\mathbf{S}) = H(\varepsilon_p), \tag{2.1}$$

where

$$\varepsilon_p = \sqrt{\left(\frac{2}{3}\right)} \int_0^t \sqrt{[tr(\dot{\mathbf{E}}\dot{\mathbf{E}})]}\, \mathrm{d}\tau$$

is the so-called equivalent strain, depending on the strain-path and the total plastic distortion (see Hill, [2], Chapter II, p. 3). $H(\varepsilon_p)$ is supposed to be a monotonically increasing positive function of its argument.

Then, observing that on the yield locus and with the von Mises criterion:

$$\mathbf{S} = \sqrt{\left(\frac{2}{3}\right)} \frac{H}{\sqrt{[tr(\dot{\mathbf{E}}\dot{\mathbf{E}})]}} \dot{\mathbf{E}}, \tag{2.2}$$

we obtain the plastic power density:

$$tr(\mathbf{S}\dot{\mathbf{E}}) = \sqrt{\left(\frac{2}{3}\right)} H \sqrt{[tr(\dot{\mathbf{E}}\dot{\mathbf{E}})]}.$$

The total plastic power is given by the functional:

$$\Phi(\mathbf{v}) = \int_V tr(\mathbf{S}\dot{\mathbf{E}})\, \mathrm{d}V. \tag{2.3}$$

We denote again by $\langle \mathbf{T}, \mathbf{v} \rangle$ the power of the external loads, according to the equation

$$\langle \mathbf{T}, \mathbf{v} \rangle = \int_{B_2} \mathbf{T}\mathbf{v}\, \mathrm{d}S,$$

and finally by $\|\mathbf{v}\|$ the norm of $\mathbf{v}$, that is:

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle.$$

If the external loads **T** depend proportionally according to a parameter λ on an assigned distribution $\mathbf{T}_0$, it is well known (see Prager and Hodge [12]) that the value of λ which determines the beginning of the plastic flow is the solution of the following minimum problem:

$$\lambda = \min \frac{\Phi(\mathbf{v}^*)}{\langle \mathbf{T}_0, \mathbf{v}^* \rangle}, \tag{2.4}$$

in the class of $\dot{\mathbf{E}}^*$ kinematically admissible functions, that is deriving from a velocity field $\mathbf{v}^*$, which satisfies the boundary conditions on B_1, such that $tr(\dot{\mathbf{E}}^*) = 0$ (incompressibility condition) and such that $\langle \mathbf{T}_0, \mathbf{v}^* \rangle > 0$.

A further formulation which is frequently more convenient is that of minimizing the functional $\Phi(\mathbf{v}^*)$ under the additional condition

$$\langle \mathbf{T}_0, \mathbf{v}^* \rangle = 1 \tag{2.5}$$

in the same class of functions.

We successively demonstrate the existence of a minimum for the problem (2.4), specifying in which class of functions and under which conditions this research must be placed

(Section 3); we examine the method of constructing the minimizing sequences and of estimating the error as to the exact solution (Sections 4 and 5); we discuss some methods to solve numerically the non-linear problem, supplying a numerical example (Section 7).

3. EXISTENCE OF THE MINIMUM

As one proceeds in the linear problems (see Mikhlin [9], p. 3) we consider the real complete† Hilbert space of square summable functions in V endowed with the norm:

$$|\mathbf{v}|^2 = \int_V tr(\dot{\mathbf{E}}\dot{\mathbf{E}})\,\mathrm{d}V, \tag{3.1}$$

which is also called the energy norm. For the elements of this space which are kinematically admissible the inequality

$$|\mathbf{v}| \geq \gamma\|\mathbf{v}\|, \tag{3.2}$$

where γ is a positive constant, must be valid.

As Mikhlin [9] has demonstrated, if the non-linear functional $\Phi(\mathbf{v})$ can be placed in the form:

$$\Phi(\mathbf{v}) = \int_V \mathrm{d}v \int_0^{tr(\dot{\mathbf{E}}\dot{\mathbf{E}})} \rho(\xi)\,\mathrm{d}\xi \tag{3.3}$$

where in the interval $0 \leq \xi < \infty$ the inequality $\rho(\xi) \geq \rho_0 > 0$ is valid, then the related inequality,

$$\Phi(\mathbf{v}) \geq \rho_0|\mathbf{v}|^2, \tag{3.4}$$

assures that:

(*a*) the functional $\Phi(\mathbf{v})$ is bounded from below;

(*b*) every minimizing sequence converges in the metric (3.1) to some limit;

(*c*) the convergence in energy implies the convergence in norm.

Now, the equivalency of the two forms (2.3) and (3.3) is assured putting:

$$\rho(\xi) = \frac{1}{2}\sqrt{\left(\frac{2}{3}\right)}\left(\dot{H} + \frac{H}{\sqrt{(\xi)}}\right) \tag{3.5}$$

where $\dot{H}$ denotes the derivative of H with respect to the variable $\sqrt{[tr(\dot{\mathbf{E}}\dot{\mathbf{E}})]}$. But, among the usual laws of hardening, $\dot{H}$ has a minimum H_0‡, whence, assuming

$$\rho_0 = \rho_{\min}(\xi) = \tfrac{1}{2}\sqrt{(\tfrac{2}{3})}\dot{H}_0,$$

the validity of (3.4) is demonstrated. From this consideration derives also that $\rho(\xi)$ is a monotonically decreasing function of its argument.

† If the space is not complete it is necessary to complete it (see Weinberger [14]).

‡ More precisely we can write:

$$\dot{H} = \mathrm{tg}\,\beta_p \frac{\mathrm{d}\varepsilon_p}{\mathrm{d}\sqrt{[tr(\dot{\mathbf{E}}\dot{\mathbf{E}})]}}$$

where tg β_p is the plastic modulus of a uniaxial state of comparison (Reckling [13], Chapter III, p. 9).

4. THE RITZ METHOD

The Ritz method is used for constructing a minimal sequence for $\Phi(\mathbf{v})$. For this purpose we select a sequence of elements

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \tag{4.1}$$

satisfying the two conditions:

(a) for any n, the elements are linearly independent;

(b) the coordinate system is complete and possibly orthonormal in energy;

and we give an approximate solution $\mathbf{v}_n$ of problem (2.4) in the form:

$$\mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{u}_k \tag{4.2}$$

where the α_k are constants selected so that $\Phi(\mathbf{v}_n)$ is a minimum with respect to the parameters α_k, with the additional condition:

$$\langle \mathbf{T}_0, \mathbf{v}_n \rangle = 1. \tag{4.3}$$

The solution of this problem is an algebraic matter leading to a non-linear Ritz system treatable for instance by the iterative Kachanov method (Mikhlin [10], 3, 10.4).

The calculation of the coefficients α_k may be effected considering the functional:

$$\Phi(\mathbf{v}_n) - \lambda_n[\langle \mathbf{T}_0, \mathbf{v}_n \rangle - 1], \tag{4.4}$$

where λ_n is a suitable Lagrange multiplier. By integration with respect to the coordinate function $\mathbf{u}_k$, we obtain from (4.4) the function:

$$\Psi(\boldsymbol{\alpha}) - \lambda_n(\boldsymbol{\alpha} \,.\, \mathbf{T} - 1), \tag{4.5}$$

having denoted by the vector $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ the set of the independent variables and by $\mathbf{T} \equiv (T_1, \ldots, T_n)$ the vector with components $T_1 = \langle \mathbf{T}_0, \mathbf{u}_1 \rangle, \ldots, T_n = \langle \mathbf{T}_0, \mathbf{u}_n \rangle$. The explicit form of $\Psi(\boldsymbol{\alpha})$ is given by (3.3), that is:

$$\Psi(\boldsymbol{\alpha}) = \int_V \mathrm{d}v \int_0^{\sum_{i,j=1}^{n} \alpha_i \alpha_j \, tr(\dot{\mathbf{E}}_i \dot{\mathbf{E}}_j)} \rho(\xi)\,\mathrm{d}\xi, \tag{4.6}$$

where $\dot{\mathbf{E}}_i$ is the deformation rate tensor associated with $\mathbf{u}_i$. Setting the partial derivatives of (4.5) equal to zero, we derive the non-linear system:

$$\operatorname{grad} \Psi(\boldsymbol{\alpha}) - \lambda_n \mathbf{T} = 0 \quad \text{or} \quad \Psi_{,k}(\boldsymbol{\alpha}) - \lambda_n T_k = 0 \qquad (k = 1, \ldots, n), \tag{4.7}$$

which, with the normalizing condition:

$$\boldsymbol{\alpha} \,.\, \mathbf{T} = 1 \quad \text{or} \quad \sum_{k=1}^{n} \alpha_k T_k = 1,$$

permits the calculation of $\boldsymbol{\alpha}$ and λ_n.

We can prove that $\boldsymbol{\alpha}$, so constructed, is an effective minimizing sequence for $\Phi(\mathbf{v})$.

Monotonicity

Increasing the components of $\boldsymbol{\alpha}$ the minimum of $\Psi(\boldsymbol{\alpha})$ cannot increase; therefore the sequence of $\Psi(\boldsymbol{\alpha})$ is monotone and non-increasing with n(Monotonicity principle of Weinberger [15]).

Convergence

If $\mathbf{v}_0$ is the exact solution minimizing $\Phi(\mathbf{v})$, it is possible to choose such an index n and a vector $\mathbf{w}_n$ that the inequality

$$\Phi(\mathbf{w}_n) - \Phi(\mathbf{v}_0) < \varepsilon \tag{4.8}$$

is valid. In fact from:

$$\Phi(\mathbf{w}_n) - \Phi(\mathbf{v}_0) = \int_V \mathrm{d}v \int_{tr(\dot{\mathbf{E}}_0\dot{\mathbf{E}}_0)}^{tr(\dot{\mathbf{E}}_n\dot{\mathbf{E}}_n)} \rho(\xi)\,\mathrm{d}\xi,$$

applying the mean value theorem we deduce:

$$\int_V \mathrm{d}V \int_{tr(\dot{\mathbf{E}}_0\dot{\mathbf{E}}_0)}^{tr(\dot{\mathbf{E}}_n\dot{\mathbf{E}}_n)} \rho(\xi)\,\mathrm{d}\xi \le \int_V \rho^*|tr(\dot{\mathbf{E}}_n\dot{\mathbf{E}}_n) - tr(\dot{\mathbf{E}}_0\dot{\mathbf{E}}_0)|\,\mathrm{d}V \tag{4.9}$$

where ρ^* is the maximum of ρ between $tr(\dot{\mathbf{E}}_0\dot{\mathbf{E}}_0)$ and $tr(\dot{\mathbf{E}}_n\dot{\mathbf{E}}_n)$, whence by the decomposition:

$$tr(\dot{\mathbf{E}}_n\dot{\mathbf{E}}_n) - tr(\dot{\mathbf{E}}_0\dot{\mathbf{E}}_0) = tr[(\dot{\mathbf{E}}_n + \dot{\mathbf{E}}_0)(\dot{\mathbf{E}}_n - \dot{\mathbf{E}}_0)],$$

and Schwartz's inequality, it follows:

$$\Phi(\mathbf{w}_n) - \Phi(\mathbf{v}_0) \le K(\mathbf{v}_0, \mathbf{w}_n)|\mathbf{w}_n - \mathbf{v}_0|$$

with

$$K(\mathbf{v}_0, \mathbf{w}_n) = \sqrt{\left\{\int_V \rho^{*2} tr[(\dot{\mathbf{E}}_n + \dot{\mathbf{E}}_0)(\dot{\mathbf{E}}_n + \dot{\mathbf{E}}_0)]\,\mathrm{d}V\right\}}.$$

But, as the system $\mathbf{u}_k$ is complete in energy, from $|\mathbf{w}_n - \mathbf{v}_0| \to 0$ and $K(\mathbf{w}_n, \mathbf{v}_0)$ is bounded†, inequality (4.8) holds (Mikhlin [7], Section 14). On the other hand, by the way of constructing $\Phi(\mathbf{v}_n)$, as a minimum among all the n-dimensional functionals, we have $\Phi(\mathbf{v}_n) \le \Phi(\mathbf{w}_n)$ and whence $\Phi(\mathbf{v}_n) \to \Phi(\mathbf{v}_0)$.

Stability

By immediate extension of the notion of stability for the linear Ritz system (Mikhlin [8], [9], 7.4), we can say that the non-linear system (4.7) is stable with respect to small variations in its coefficients and free terms, if these determine changes of the same order in the solutions α_k. This propriety is assured if the smallest eigenvalue of the matrix

$$R_n = \mathrm{grad}^{(2)}\,\Psi(\boldsymbol{\alpha})$$

is bounded from below by a positive number independent of n. But such a condition is implied in equation (3.4), whence moreover one can deduce the inequality

$$\langle R_n\boldsymbol{\alpha}, \boldsymbol{\alpha}\rangle \ge 2\rho_0\|\boldsymbol{\alpha}\|^2, \tag{4.10}$$

and, therefore, that $2\rho_0$ is a lower bound, independent of n, for the first norm of R_n.‡

†We can easily verify that $K(\mathbf{w}_n, \mathbf{v}_0)$ is bounded in the class of functions with finite energy.

‡ In fact, from inequality (3.4) written for $\mathbf{v}_n = \Sigma_1^n \alpha_k \mathbf{u}_k$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \varepsilon\boldsymbol{\alpha}_1$, where ε is an infinitesimal parameter, we obtain:

$$\Psi(\boldsymbol{\alpha} + \varepsilon\boldsymbol{\alpha}_1) \ge \rho_0\|\boldsymbol{\alpha}_0 + \varepsilon\boldsymbol{\alpha}_1\|^2,$$

whence expanding the left-hand side in Taylor's series and comparing the terms of the same order in ε we derive (4.10).

5. DEGREE OF CONVERGENCE (MIKHLIN [10], 5)

The preceding results allow us to make an estimate of the order of accuracy of the approximate solutions when only the first n terms of series (4.2) are taken into account. In fact, if $\boldsymbol{\alpha}$ is an n-dimensional vector satisfying equation (4.7), and the normality condition:

$$\sum_{k=1}^{n} \alpha_k T_k = 1,$$

an upper bound of the absolute minimum is given by:

$$\lambda_n = \Psi(\boldsymbol{\alpha}). \tag{5.1}$$

Now, for completeness, the absolute minimum λ_0 can be expressed by $\Psi(\boldsymbol{\alpha}_0)$, where $\boldsymbol{\alpha}_0$ is a suitable infinite-dimensional vector normalized by the following condition:

$$\sum_{k=1}^{\infty} \alpha_{0k} T_k = 1.$$

But as $\Psi(\boldsymbol{\alpha})$ is a convex function of its argument and $\Psi(\boldsymbol{\alpha}_0)$ its minimum, the inequality:

$$\Psi(\boldsymbol{\alpha}) - \Psi(\boldsymbol{\alpha}_0) \leq \text{grad}\, \Psi(\boldsymbol{\alpha}) \,.\, (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0),$$

or equivalently

$$\lambda_n - \lambda_0 \leq \sum_{k=1}^{n} \Psi_{,k}(\boldsymbol{\alpha})(\alpha_k - \alpha_{0k}), \tag{5.2}$$

must be valid. Moreover equation (4.7) gives:

$$\Psi_{,k}(\boldsymbol{\alpha}) = \lambda_n T_k \qquad (k = 1, \ldots, n),$$

therefore the inequality (5.2) permits us to write:

$$|\lambda_n - \lambda_0| \leq |\lambda_n| \left| \sum_{k=1}^{n} T_k(\alpha_k - \alpha_{0k}) \right| = |\lambda_n| \left| \sum_{k=n+1}^{\infty} \alpha_{0k} T_k \right|. \tag{5.3}$$

From this last inequality derives, beside the convergence, also an estimate of the nth approximation, if it is possible to bound the Fourier coefficients α_{0k}, as it is often the case.†

6. PERFECTLY RIGID–PLASTIC SOLIDS

Some special considerations have to be introduced for rigid–perfectly plastic solids, because the hardening factor $\text{tg}\beta_p$ is zero, whence ρ_0 is zero and inequality (3.2) is no longer valid. Now, the functional (3.3) is not coercive and the convergence in energy does not ensure the convergence in norm. Also the application of the Ritz method is no longer possible, because the functional (3.3) may not be convex.

However, the two principal circumstances that justify the reduction to finite dimension of the problem remain valid; they are *monotonicity* and *convergence*, the former depending on the increase of dimension, the latter on an inequality of the type (4.9). Of course, the

† See e.g. Kantorovich and Krylov [4] I.5.

uniqueness of a minimizing solution cannot be expected (see Koiter, [5]). This degenerate case requires the particular device of considering the perfectly rigid–plastic material as a limiting model of a rigid-work-hardening material.†

7. EXAMPLE: HOOKE–COULOMB MODEL

An illustration of the previous results is given by the limit analysis of the mechanical model with two degrees of freedom depicted in Fig. 1. The solid M can translate on its

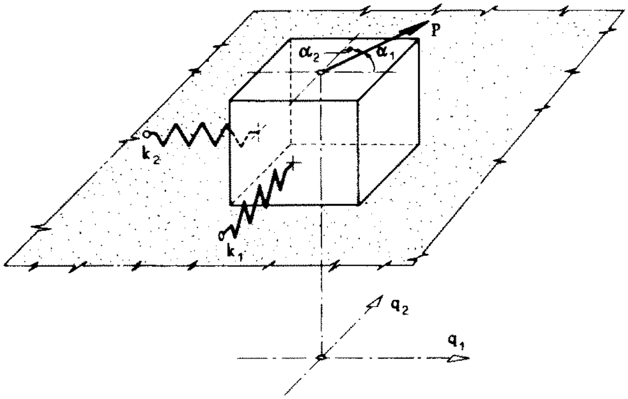

FIG. 1.

supporting plane when the force P overcomes a limit value S_0. After the beginning of motion the rates of displacement q_1, q_2 are constrained by the springs with elastic constants k_1, k_2. The plastic power is expressed by‡:

$$\Phi(\mathbf{q}) = S_0\sqrt{(q_1^2+q_2^2)}+\tfrac{1}{2}(k_1q_1^2+k_2q_2^2), \tag{7.1}$$

while the power of the external load P, put equal to 1, is $\alpha_1q_1+\alpha_2q_2$, where α_1, α_2 are the direction cosines of P. Problem (2.4) can now be formulated as:

$$\Phi(\mathbf{q}) = \min, \tag{7.2}$$

under the condition:

$$\boldsymbol{\alpha}\mathbf{q} = \alpha_1q_1+\alpha_2q_2 = 1. \tag{7.3}$$

This is a problem of the ordinary minimum and so the existence of a unique solution is warranted by inequality:

$$|\mathbf{q}|^2 = k_1q_1^2+k_2q_2^2 \geq \gamma(q_1^2+q_2^2) = \gamma\|\mathbf{q}\|^2 \tag{7.4}$$

where γ is $\min(k_1, k_2)$, and by inequality:

$$\Phi(\mathbf{q}) = \frac{1}{2}\int_0^{|\mathbf{q}|^2}\left(1+\frac{S_0\|\mathbf{q}\|}{\sqrt{(\xi)}|\mathbf{q}|}\right)\mathrm{d}\xi \geq \rho_0|\mathbf{q}|^2 \tag{7.5}$$

with $\rho_0 = \frac{1}{2}$.

† This way of formulating the problem is the mechanical equivalent of the so-called "elliptical regularization" for non-coercive functionals (see Lions and Stampacchia [6]). Analogous techniques are also applied by Fox [1].
‡ We have denoted by $\mathbf{q}$ the vector (q_1, q_2).

To follow the scheme of the Ritz method, we approximate the exact solution in two ways:

I. Choosing $\mathbf{q}^{(1)} = (q_1, 0)$ as approximation of the effective configuration of collapse, the substitution in (7.2), (7.3) gives immediately:

$$\Phi(\mathbf{q}^{(1)}) = \frac{S_0}{|\alpha_1|} + \frac{1}{2\alpha_1^2} k_1 \tag{7.6}$$

which is of course an upper bound of the true collapse load.

II. Putting now $\mathbf{q}^{(2)} = (q_1, q_2)$ we minimize (7.2) under condition (7.1) by the above-mentioned Kackanov method (Mikhlin [10], 3, 10.4). The calculation is made in two steps:

(1) Assuming in (7.5) $\rho(\xi) = \rho_0 = \frac{1}{2}$, the functional (7.5) is reduced to a quadratic one, whose constrained minimum is reached for:

$$q_1 = \frac{\alpha_1/k_1}{(\alpha_1^2/k_1)+(\alpha_2^2/k_2)}, \qquad q_2 = \frac{\alpha_2/k_2}{(\alpha_1^2/k_1)+(\alpha_2^2/k_2)} \tag{7.7}$$

(2) We set afterwards in (7.5) $\rho(\xi) = \rho_1 = \frac{1}{2} + S_0\|\mathbf{q}^{(2)}\|/|\mathbf{q}^{(2)}|$, where $\mathbf{q}^{(2)}$ has components (7.7), and we calculate the minimum of this new quadratic functional. Since from (7.7) we derive:

$$|\mathbf{q}^{(2)}|^2 = \left(\frac{\alpha_1^2}{k_1} + \frac{\alpha_2^2}{k_2}\right)^{-1}, \qquad \|\mathbf{q}^{(2)}\|^2 = \left(\frac{\alpha_1^2}{k_1} + \frac{\alpha_2^2}{k_2}\right)^{-1} \sqrt{\left(\frac{\alpha_1^2}{k_1^2} + \frac{\alpha_2^2}{k_2^2}\right)},$$

the minimum of the functional so modified is attained for the same pair (7.7) and the equation holds:

$$\Phi(\mathbf{q}^{(2)}) = \left[\frac{1}{2} + S_0 \sqrt{\left(\frac{\alpha_1^2}{k_1^2} + \frac{\alpha_2^2}{k_2^2}\right)}\right] \left(\frac{\alpha_1^2}{k_1} + \frac{\alpha_2^2}{k_2}\right)^{-1}. \tag{7.8}$$

Remark I. The stability of the numerical solution of steps (1), (2) follows at once from condition (4.10), because the smallest eigenvalue of the matrix:

$$\operatorname{grad}^{(2)} \Phi(\mathbf{q}) = \begin{bmatrix} k_1 + S_0\left(\dfrac{1}{\sqrt{(q_1^2+q_2^2)}} - \dfrac{q_1^2}{\sqrt{[(q_1^2+q_2^2)^3]}}\right) & -S_0 \dfrac{q_1 q_2}{\sqrt{[(q_1^2+q_2^2)^3]}} \\ -S_0 \dfrac{q_1 q_2}{\sqrt{[(q_1^2+q_2^2)^3]}} & k_2 + S_0\left(\dfrac{1}{\sqrt{(q_1^2+q_2^2)}} - \dfrac{q_2^2}{\sqrt{[(q_1^2+q_2^2)^3]}}\right) \end{bmatrix}$$

is bounded from below by $\gamma = \min(k_1, k_2)$.

Remark II. As far as the convergence of the approximate solutions is concerned, we apply formula (5.3) to the first trial function of the Ritz method. Since the second approximation gives the true minimum, all terms in (5.3) are known, and we can write:

$$|\lambda_1 - \lambda_0| \leq |\lambda_1||\alpha_{02} T_2|, \tag{7.9}$$

where λ_1 is given by:

$$\lambda_1 = 1 + 2S_0\left(\frac{\alpha_1^2}{k_1^2} + \frac{\alpha_2^2}{k_2^2}\right)\left(\frac{\alpha_1^2}{k_1} + \frac{\alpha_2^2}{k_2}\right)^{-1}$$

is the Lagrangian multiplyer of problem I; moreover, since $\alpha_{02} = q_2$, from the second equality of (7.7) and $T_2 = \alpha_2$, we obtain

$$|\lambda_1-\lambda_0| \le \left[1+2S_0\left(\frac{\alpha_1^2}{k_1^2}+\frac{\alpha_2^2}{k_2^2}\right)\left(\frac{\alpha_1^2}{k_1}+\frac{\alpha_2^2}{k_2}\right)^{-1}\right]\frac{\alpha_1^2/k_2}{(\alpha_1^2/k_1)+(\alpha_2^2/k_2)},$$

as an upper bound of the error of the first approximation.

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Абстракт—Расширяются некоторые фундаментальные теоремы существования и однозначности для минимума нелинейных функционалов в теории несущей способности. Затем исследуется сходимостъ метода Ритца, и в особенности: а/ условия, при которых приближения дают доведенную к минимуму последователъностъ; б/ устойчивость численного метода; в/ оценку степени сходимости. Пример конечно-размерной системы иллюстрирует основные резулътаты.